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Enhancing synchronism of chaotic systems

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Recent work has considered the situation where a state variable (or variables) of a chaotically evolving system is used as an input to a replica of part of the original system. It was found that the replica subsystem often synchronizes to the chaotic evolution of the original system, and it has been suggested that this phenomenon may be used for secure communications. In this paper we point out that exact synchronism may also occur for a large class of systems that are not replicas of part of the original system. This allows greater freedom in choosing synchronizer systems, and we discuss the possibility of using this freedom to choose synchronizer systems with improved performance. Two explicit examples illustrating this statement are given, one where the chaotic system consists of three autonomous differential equations, and the other where the chaotic system is a two-dimensional map.

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Recently, Pecora and Carroll [1] have studied the situation where a state variable (or variables) of a chaotic system is used as an input to drive a subsystem that is a replica of part of the original system. They find that the replica subsystem sometimes synchronizes to the chaotic evolution of the original system. The condition for this synchronism to occur depends on the original chaotic system and on the choice of the part of the original system that is used as the replica subsystem. A particularly intriguing aspect of chaos synchronism is its possible application to achieving secure communications (see, for example, [2] and references therein). We list other related works in [3].

Consider a chaotic system, $d\mathbf{Z}/dt=\mathbf{F}(\mathbf{Z})$, where $\mathbf{Z}$ is an m-dimensional vector. Divide the m state variables into two classes via

$$\mathbf{Z}=\begin{bmatrix}\mathbf{x}\\ \mathbf{y}\end{bmatrix},$$

where $\mathbf{x}$ is m_1 dimensional and $\mathbf{y}$ is m_2 dimensional, with $m_1+m_2=m$ (usually $m_1=1$ [1–3]). We refer to $\mathbf{x}$ as the input or drive. Writing the original dynamical system as

$$d\mathbf{x}/dt=\mathbf{G}(\mathbf{x},\mathbf{y}) , \tag{1}$$

$$d\mathbf{y}/dt=\mathbf{H}(\mathbf{x},\mathbf{y}) , \tag{2}$$

where

$$\mathbf{F}(\mathbf{Z})=\begin{bmatrix}\mathbf{G}(\mathbf{x},\mathbf{y})\\ \mathbf{H}(\mathbf{x},\mathbf{y})\end{bmatrix},$$

the driven replica subsystem is written as

$$d\hat{\mathbf{y}}/dt=\mathbf{H}(\mathbf{x},\hat{\mathbf{y}}) . \tag{3}$$

Here we call $\hat{\mathbf{y}}$ the subsystem response, and, in what follows, we use a superscribed circumflex to denote a response variable. We say the subsystem (3) synchronizes to the chaotic evolution of (1) and (2), if

$$\lim_{t\to\infty}|\mathbf{y}(t)-\hat{\mathbf{y}}(t)|=0 , \tag{4}$$

for typical $\mathbf{y}(0)\neq\hat{\mathbf{y}}(0)$.

In the context of communication we can imagine that the state variables in the drive vector $\mathbf{x}$ are transmitted from site A to site B. The full state $\mathbf{Z}$ of the system at A is unknown at B, because, although we know $\mathbf{x}(t)$, we do not know $\mathbf{y}(t)$. By feeding the known signal $\mathbf{x}(t)$ into a replica response subsystem (3) located at B, we can then obtain $\mathbf{y}(t)$, if the replica synchronizes with the original system [see (4)], and if we wait long enough for $\hat{\mathbf{y}}(t)$ to closely approach $\mathbf{y}(t)$.

The occurrence of this synchronism is conditioned on whether the largest subsystem Lyapunov exponent is negative [1]. Consider infinitesmal deviations of $\hat{\mathbf{y}}$ from $\mathbf{y}$, i.e.,

$$\hat{\mathbf{y}}(t)=\mathbf{y}(t)+\delta\hat{\mathbf{y}}(t) .$$

From (3)

$$d\delta\hat{\mathbf{y}}/dt=\delta\hat{\mathbf{y}}\cdot\partial\mathbf{H}(\mathbf{x},\hat{\mathbf{y}})/\partial\hat{\mathbf{y}}|_{\hat{\mathbf{y}}=\mathbf{y}} , \tag{5}$$

where $\mathbf{x}(t)$ and $\mathbf{y}(t)$ are solutions of (1) and (2). The largest subsystem Lyapunov exponent, denoted $\hat{\Lambda}$, is then given by solving (5) using a typical orbit $(\mathbf{x}(t),\mathbf{y}(t))$ for the original system (1) and (2) and a typical choice for the orientation of $\delta\hat{\mathbf{y}}(0)$,

$$\hat{\Lambda}=\lim_{t\to\infty}\frac{1}{t}\ln\frac{|\delta\hat{\mathbf{y}}(t)|}{|\delta\hat{\mathbf{y}}(0)|} . \tag{6}$$

The point of this paper is that it may be advantageous to use a subsystem that is not a replica of part of the original system. In particular, we note that synchronism is also possible if, instead of the replica system (3), we uti-

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lize a nonreplica system of the form

$$d\hat{\mathbf{x}}/dt=\mathbf{K}(\mathbf{x},\hat{\mathbf{x}},\hat{\mathbf{y}}) , \quad (7)$$

$$d\hat{\mathbf{y}}/dt=\mathbf{L}(\mathbf{x},\hat{\mathbf{x}},\hat{\mathbf{y}}) , \quad (8)$$

provided that the functions **K** and **L** satisfy the conditions

$$\mathbf{K}(\mathbf{x},\mathbf{x},\mathbf{y})=\mathbf{G}(\mathbf{x},\mathbf{y}) , \quad (9)$$

$$\mathbf{L}(\mathbf{x},\mathbf{x},\mathbf{y})=\mathbf{H}(\mathbf{x},\mathbf{y}) . \quad (10)$$

We now view the vector,

$$\hat{\mathbf{Z}}(t)=\begin{bmatrix}\hat{\mathbf{x}}(t)\\ \hat{\mathbf{y}}(t)\end{bmatrix} ,$$

as the "response."

Synchronism, $\hat{\mathbf{Z}}(t)=\mathbf{Z}(t)$, then represents a possible solution of (7) and (8). Whether the synchronizing solution is stable depends on the largest Lyapunov exponent for the synchronizer system (7) and (8), which is obtained from

$$d\delta\hat{\mathbf{x}}/dt=\delta\hat{\mathbf{x}}\cdot\nabla_{\hat{\mathbf{x}}}\mathbf{K}(\mathbf{x},\hat{\mathbf{x}},\mathbf{y})|_{\hat{\mathbf{x}}=\mathbf{x}} +\delta\hat{\mathbf{y}}\cdot\nabla_{\hat{\mathbf{y}}}\mathbf{K}(\mathbf{x},\mathbf{x},\hat{\mathbf{y}})|_{\hat{\mathbf{y}}=\mathbf{y}} , \quad (11)$$

$$d\delta\hat{\mathbf{y}}/dt=\delta\hat{\mathbf{x}}\cdot\nabla_{\hat{\mathbf{x}}}\mathbf{L}(\mathbf{x},\hat{\mathbf{x}},\mathbf{y})|_{\hat{\mathbf{x}}=x} +\delta\hat{\mathbf{y}}\cdot\nabla_{\hat{\mathbf{y}}}\mathbf{L}(\mathbf{x},\mathbf{x},\hat{\mathbf{y}})|_{\hat{\mathbf{y}}=y} , \quad (12)$$

with $\delta\hat{\mathbf{y}}$ in (6) replaced by

$$\delta\hat{\mathbf{Z}}(t)=\begin{bmatrix}\delta\hat{\mathbf{x}}(t)\\ \delta\hat{\mathbf{y}}(t)\end{bmatrix} .$$

The idea is that there are an infinite number of functions **K** and **L** satisfying (9) and (10) for given **G** and **H**. Thus one is not necessarily constrained to the use of a replica (3) when choosing a system to achieve synchronism. This leads to considerably more flexibility in applications. [We imagine that the original chaotic system, (1) and (2), with the choice of the drive **x** is fixed by the given application.]

We speculate that this added flexibility may facilitate potential improvement in synchronism. Some such improvements might include the following: (1) achieving synchronism when a replica subsystem does not synchronize (i.e., $\hat{\Lambda}>0$ for the replica subsystem); (2) enabling faster convergence to the synchronized state; (3) eliminating or reducing the size of spurious subsystem basins of attraction in which the subsystem does not synchronize; and (4) improving the performance of signal recovery techniques for situations where the chaotic time series is used to mask a small information bearing signal [2].

In what follows we present two examples to illustrate points (1) and (2) for enhancing chaos synchronism mentioned above. The first example is a differential equation system (the Rössler equations) that we investigate numerically. The second example concerns a simple chaotic map that can be solved exactly.

Example 1. The Rössler system we treat in this example can be written as the following set of ordinary differential equations:

$$dx/dt=b+x(y_1-c)\equiv G(x,y_1,y_2) , \quad (13)$$

$$dy_1/dt=-x-y_2\equiv H_1(x,y_1,y_2) , \quad (14)$$

$$dy_2/dt=y_1+ay_2\equiv H_2(x,y_1,y_2) . \quad (15)$$

For $a=0.398$, $b=2$, and $c=4$ (also used in the subsequent calculations), the above system has a chaotic attractor.

Suppose that in a given application we are constrained to transmit only the x component of the above equation with the values a, b, and c as given above. The task is then to attempt obtaining synchronism using this x as a drive. Consider the replica subsystem,

$$d\hat{y}_1/dt=-x-\hat{y}_2 , \quad (16)$$

$$d\hat{y}_2/dt=\hat{y}_1+a\hat{y}_2 . \quad (17)$$

The largest subsystem Lyapunov exponent, assuming $a^2<4$, is

$$\hat{\Lambda}=a/2 . \quad (18)$$

For $a>0$, we have $\hat{\Lambda}>0$, and thus synchronism is not achieved for the parameter values chosen above. Figure 1 shows the times series $y_1(t)$ from the original system plotted together with $\hat{y}_1(t)$ of the response. As we can see, $\hat{y}_1(t)$ diverges from the synchronizing solution as time increases. Here we have chosen, in the numerical calculation, the initial condition for the replica subsystem to be slightly different from that of the original system.

Now we ask, with the same transmitted signal x as a drive, can we achieve synchronism by using a nonreplica response system,

$$d\hat{x}/dt=K(x,\hat{x},\hat{y}_1,\hat{y}_2) , \quad (19)$$

$$d\hat{y}_1/dt=L_1(x,\hat{x},\hat{y}_1,\hat{y}_2) , \quad (20)$$

$$d\hat{y}_2/dt=L_2(x,\hat{x},\hat{y}_1,\hat{y}_2) , \quad (21)$$

with K, L_1, and L_2 satisfying (9) and (10)? We explore this question numerically by taking, for simplicity,

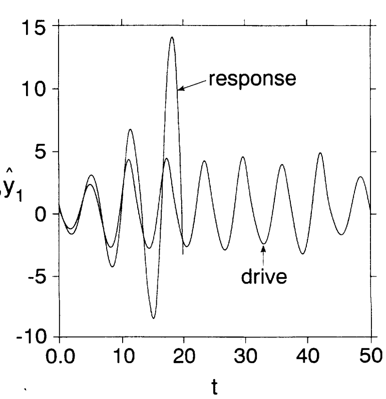

FIG. 1. Trajectories showing the replica subsystem response [(16),(17)] diverging from the solution of the original Rössler system [(13)–(15)].

$$K(x,\hat{x},\hat{y}_1,\hat{y}_2)=G(\hat{x},\hat{y}_1,\hat{y}_2)+\alpha(\hat{x}-x)\ ,$$

$$L_1(x,\hat{x},\hat{y}_1,\hat{y}_2)=H_1(\hat{x},\hat{y}_1,\hat{y}_2)+\beta(\hat{x}-x)\ ,$$

$$L_2(x,\hat{x},\hat{y}_1,\hat{y}_2)=H_2(\hat{x},\hat{y}_1,\hat{y}_2)+\gamma(\hat{x}-x)\ ,$$

which meet the conditions (9) and (10). [We remark that this choice of nonreplica systems, while sufficient for the purpose of this paper, is somewhat arbitrary. Other choices, such as those including terms depending on $\hat{y}_1$ and $\hat{y}_2$, and nonlinearly on $(\hat{x}-x)$, can be made in response to the particular need of the problems at hand.] The new response system is then

$$d\hat{x}/dt=b+x(\hat{y}_1-c)+\alpha(\hat{x}-x)\ , \tag{22}$$

$$d\hat{y}_1/dt=-x-\hat{y}_2+\beta(\hat{x}-x)\ , \tag{23}$$

$$d\hat{y}_2/dt=\hat{y}_1+a\hat{y}_2+\gamma(\hat{x}-x)\ . \tag{24}$$

Greater flexibility in achieving improved performance for this system is rendered by the freedom in choosing the values of the arbitrary parameters α, β, and γ.

Consider a line through the origin in the parameter space spanned by α, β, and γ

$$\alpha/\alpha_e=\beta/\beta_e=\gamma/\gamma_e\ . \tag{25}$$

Here we choose $\alpha_e=-3$, $\beta_e=1$, and $\gamma_e=-5$. In Fig. 2, we plot the numerically calculated largest Lyapunov exponent $\hat{\Lambda}$ for the response system (22), (23), and (24) as a function of β, with α and γ varying according to (25). The value of $\hat{\Lambda}$ generally decreases as β increases, and becomes negative for $\beta>0.27$, indicating the onset of synchronism. Figure 3 shows y_1 and $\hat{y}_1$ calculated using the same initial conditions as that in Fig. 1 but for $\beta=0.7$ ($\hat{\Lambda}=-0.25$). Thus we obtain synchronism using a nonreplica response system for a case where a replica response system does not synchronize.

As mentioned earlier, another possible advantage of using nonreplica response systems is to improve the convergence to the synchronized solution. To illustrate, we show, in Fig. 4, a case of slow convergence for $\beta=0.28$, which is just above the synchronism threshold. In this case, $\hat{\Lambda}=-0.006$, and the two solutions become essentially indistinguishable only for $t>200$. In contrast, as has been seen in Fig. 3, for a different choice of the response system, much faster convergence can be attained.

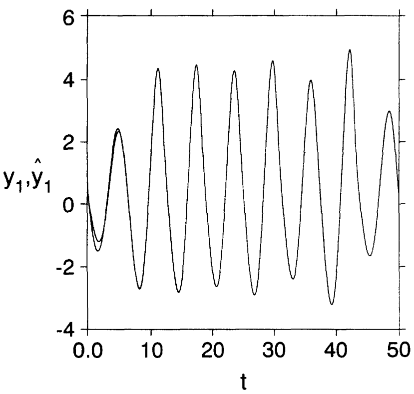

FIG. 3. Trajectories showing the rapid convergence of the nonreplica response system to the synchronizing solution for $\beta=0.7$. $\hat{\Lambda}=-0.25$. Initial conditions used here are the same as those in Fig. 1.

Example 2. In this example, we consider the following two-dimensional area-preserving map:

$$x_{n+1}=(x_n+y_n)\mathrm{mod}1\equiv G(x_n,y_n)\ , \tag{26}$$

$$y_{n+1}=[y_n+Ks(x_n+y_n)]\equiv H(x_n,y_n)\ , \tag{27}$$

where K is a parameter of the system and $s(x)$ is the sawtooth function defined as

$$s(x)\equiv(x\ \mathrm{mod}1)-\tfrac{1}{2}\ .$$

The Jacobian matrix of the system (26) and (27) is constant,

$$\mathbf{J}=\begin{bmatrix}\partial G/\partial x & \partial G/\partial y\\ \partial H/\partial x & \partial H/\partial y\end{bmatrix}=\begin{bmatrix}1 & 1\\ K & 1+K\end{bmatrix}\ . \tag{28}$$

The eigenvalues of this matrix are $[(2+K)$

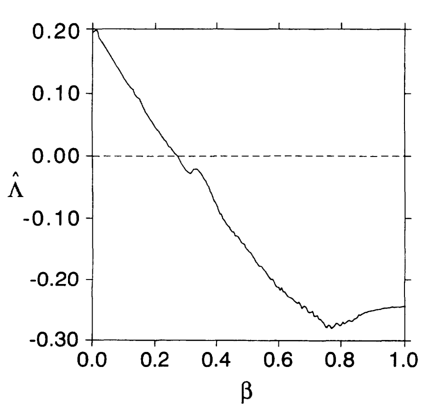

FIG. 2. Numerically evaluated largest Lyapunov exponent $\hat{\Lambda}$ as a function of β for the nonreplica response system (22)–(24).

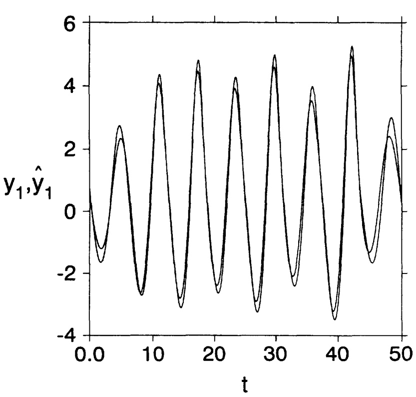

FIG. 4. Slow convergence to the synchronizing solution is encountered when the largest response Lyapunov exponent is negative but small in magnitude. $\beta=0.28$ and $\hat{\Lambda}=-0.006$. Again the same initial conditions as those in Figs. 1 and 3 are used here.

$\pm(4K+K^2)^{1/2}]/2$. If $(4K+K^2)<0$, then the eigenvalues are both complex and of magnitude 1 (no chaos). If $(4K+K^2)>0$, then the eigenvalues are both real, with one of magnitude less than 1 and one of magnitude greater than 1 (chaos). Thus the condition for chaos is $(4K+K^2)>0$, which applies if either

$$K>0\ , \tag{29}$$

or

$$K<-4\ . \tag{30}$$

Now let us consider attempting to obtain synchronism using x as the drive and the replica subsystem,

$$\hat{y}_{n+1}=H(x_n,\hat{y}_n)\ , \tag{31}$$

where $H(x,y)$ is given by (27). Since $\partial H/\partial y=1+K$, the replica subsystem Lyapunov exponent is

$$\hat{\Lambda}=\ln|1+K|\ , \tag{32}$$

which is negative only in the range

$$-2<K<0\ . \tag{33}$$

Comparing (29) and (30) with (33), we see that these conditions are mutually exclusive. That is, where the system (26) and (27) is chaotic, the subsystem (31) does not synchronize.

We now consider whether, with the same drive x_n, synchronism can be obtained by using a nonreplica response system of the form

$$\hat{x}_{n+1}=K(x_n,\hat{x}_n,\hat{y}_n)\ , \tag{34}$$

$$\hat{y}_{n+1}=L(x_n,\hat{x}_n,\hat{y}_n)\ . \tag{35}$$

For this purpose, as in the previous example, we again choose K and L to deviate from G and H by terms linear in $(\hat{x}_n-x_n)$,

$$K(x,\hat{x},\hat{y})=G(x,\hat{y})+\alpha(\hat{x}_n-x_n)\ , \tag{36}$$

$$L(x,\hat{x},\hat{y})=H(x,\hat{y})+\beta(\hat{x}_n-x_n)\ . \tag{37}$$

With this choice the Jacobian matrix of the response system is again constant,

$$\hat{\mathbf{J}}=\begin{bmatrix}\alpha & 1\\ \beta & 1+K\end{bmatrix}\ . \tag{38}$$

The eigenvalues $\hat{\lambda}_1$ and $\hat{\lambda}_2$ of $\hat{\mathbf{J}}$ satisfy

$$\lambda^2-(1+K+\alpha)\hat{\lambda}+[\alpha(1+K)-\beta]=0\ . \tag{39}$$

Writing (39) as $(\hat{\lambda}-\hat{\lambda}_1)(\hat{\lambda}-\hat{\lambda}_2)=\hat{\lambda}^2-(\hat{\lambda}_1+\hat{\lambda}_2)\hat{\lambda}+\hat{\lambda}_1\hat{\lambda}_2$, we see that $(1+K+\alpha)=\hat{\lambda}_1+\hat{\lambda}_2$ and $\alpha(1+K)-\beta=\hat{\lambda}_1\hat{\lambda}_2$, which can be solved for α and β,

$$\alpha=(\hat{\lambda}_1+\hat{\lambda}_2)-(1+K)\ , \tag{40}$$

$$\beta=[(\hat{\lambda}_1+\hat{\lambda}_2)-(1+K)](1+K)-\hat{\lambda}_1\hat{\lambda}_2\ . \tag{41}$$

Thus any choice of the response system eigenvalues $\hat{\lambda}_1$ and $\hat{\lambda}_2$ can be realized by setting the values of α and β according to (40) and (41). In particular, we can consider a case where the original system is chaotic [i.e., either (29) or (30) applies], and choose values $|\hat{\lambda}_{1,2}|<1$ yielding synchronism. In contrast, the replica subsystem (31) never synchronizes when the original system is chaotic.

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